



THE THERMODYNAMIC STRUCTURE AND CONSTITUTIVE EQUATIONS FOR FLUID- SATURATED COMPRESSIBLE AND INCOMPRESSIBLE ELASTIC POROUS SOLIDS

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Abstract—The macroscopic porous media theory consists of the mixture theory, restricted by the volume fraction concept. The incorporation of this concept leads to major difficulties in constructing a consistent macroscopic theory. With the help of thermodynamic restrictions, in addition with the multiplicative decomposition of the deformation gradient as well as experimental observations it will be shown, as to how a consistent phenomenological porous media theory for compressible and incompressible elastic porous solid models filled with an incompressible fluid can be developed which is mathematically and physically well balanced. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In order to develop a consistent phenomenological porous media theory many approaches have been carried out in the literature in the past (see, e.g., Biot and Willis, 1957; Mow *et al.*, 1980; Bowen, 1980, 1982; Passman *et al.*, 1984; de Boer, 1995a). The crucial point for the development of a consistent theory is concerned with missing field equations due to the introduction of volume fractions so that the theory is not closed. The reason is that with the introduction of the volume fraction concept to extend the macroscopic mixture theory, the microlevel is touched. For this level, however, no balance equations are provided by the mixture theory.

In order to overcome this shortcoming several attempts have been made as, e.g., the introduction of additional balance equations—more or less physically well founded—or additional evolution equations or other constitutive relations for quantities which are related to the microscale.

Another shortcoming in some existing approaches is due to the saturation condition being overlooked as a general constraint. This shortcoming leads to the consequence that no transition of compressible models to the incompressible case is possible and that no hybrid models (models containing incompressible and compressible constituents) can be developed which play a role in rock mechanics (see de Boer, 1995a). In particular, it is not possible to verify the effective stress principle. This principle states that the stress in a saturated porous solid can be additively decomposed into a weighted pore pressure part and a part which is governed by the motion of the skeleton. This principle has been experimentally and theoretically repeatedly proved (see the extensive discussion in Lade and de Boer, 1994) and should be incorporated in the porous media theory.

The goal of this paper is to construct a porous media theory for compressible and incompressible porous solids filled with an incompressible fluid that avoids the aforementioned drawbacks. A helpful tool to develop a consistent macroscopic porous media theory are restrictions for the constituents obtained from thermodynamic investigations. These investigations will be extensively discussed and a thermodynamic structure will be developed which can be used to limit the form of the constitutive equations for incompressible and compressible constituents in elastic and inelastic areas. It turns out that in the

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case of elastic constituents constitutive relations can be developed for incompressible and compressible porous solids which are mathematically and physically well balanced.

Most helpful for constructing the thermodynamic structure is to utilize the multiplicative decomposition, proposed by Bluhm and de Boer (1994), of the deformation gradient into a tensor which describes the real deformations of the real materials and into a tensor which reflects the changes of the pores. This decomposition throws light on the kinematics at the microscale and makes it easy to solve the stated closure problem. Moreover, it throws light upon the constraint due to the saturation condition. By introducing the multiplicative decomposition of the deformation gradient the rate formulation of this condition reveals that, for example, for a binary model the rates of the volume changes of the two constituents are dependent of each other at the macro- as well as the micro-level. Thus, the motions of both constituents are constrained.

With the help of the saturation condition as a constraint the effective stress principle can be clearly stated. Thus, confirming the results for incompressible media already stated in the early stage of the development of a porous media theory by Fillunger (1913, 1914, 1915) and von Terzaghi (1936) as well as the result of Suklje (1969) for a hybrid model which are carefully proved by experimental observations (see Lade and de Boer, 1994).

At this stage the contents of this paper should be briefly outlined. After the introduction of the volume fraction concept in the section of basic relations the kinematics of a general porous media model will be discussed. Then the balance equations of the mixture theory will be reviewed. The derivation of the thermodynamic structure to restrict the constitutive relations will be an essential part of this treatise. Thereafter constitutive equations for a fluid-saturated compressible and incompressible elastic porous solid will be developed. Finally, some complementary and concluding remarks will close this paper.

2. BASIC RELATIONS

In this section the material independent equations, i.e., the volume fraction concept, the kinematics, and the balance equations will be reviewed. For a complete derivation of these basic relations the reader is referred to de Boer and Ehlers (1986) and de Boer and Didwania (1995).

2.1. *The volume fraction concept*

The volume fraction concept in connection with the mixture theory has turned out to be an efficient tool to investigate saturated or empty porous solids. This concept has the effect to distribute the mass of the solid skeleton and of the fluids over the total control space shaped by the porous solid. The distribution takes place with the help of volume fraction numbers that fix the ratios of the volumes of the constituents in relation to the volume of the control space. The basic assumption for the volume fraction concept is that the pores are statistically distributed over the control space. In this case, the equality of volume and surface porosities is given as a statistical necessity. The volume fraction concept gives rise to the effect that "smeared" substitute continua for the solid and fluid phases with smooth properties arise, which fill the control space simultaneously. Thus these substitute continua show the same properties as a mixture body. Therefore the mixture theory is the adequate basis for the mechanical and thermodynamical treatment at the macroscale. However, in order to guarantee the closure of the porous media theory several requirements must be fulfilled. This problem will be discussed in Section 3.

The concept of volume fractions was introduced by Woltman (1794) (see de Boer, 1992, as well as Bluhm and de Boer, 1994). Later, the mining engineer Delesse (1848) discovered that the surface porosities are equal to the volume porosities if the pores are statistically distributed. Delesse's (1848) reflections played an important role at the very beginning of the development of a porous media theory in the first half of this century. All competent researchers in this field, namely Fillunger (1913, 1935) who called Delesse's statement the Delessian law, von Terzaghi (1934), von Terzaghi and Rendulic (1934) as well as Heinrich and Desoyer (1955, 1956) referred to Delesse's finding. In particular, Heinrich and Desoyer (1956) proved in their remarkable paper that the Delessian law is

also valid for anisotropic solids. Finally, Kubik (1979) made an interesting proposal to extend the volume fraction concept. He introduced a second order, symmetric tensor of structural permeability \mathbf{P} .

The drawback of all aforementioned suggestions is that the existence of volume fractions at the macroscale are required in each point of substitute (partial) bodies without having being properly defined. This can be done proceeding from micromechanic considerations (see de Boer and Didwania, 1995, or de Boer *et al.*, 1991). We proceed here from the geometrically averaging procedure (de Boer *et al.*, 1991) and arrive at the same result if an ensemble average, mainly used in suspension theory and the theory of fluidized beds, is applied (de Boer and Didwania, 1995). We define the volume fractions in the actual placement.

Let \mathbf{r} describe the position of a material point of a constituent with volume elements dv_μ in the control space and \mathbf{x} the centre of volume element dv . In order to determine the average volume (realistic) element dv^α of each constituent φ^α an indicator function

$$\chi^\alpha = \chi^\alpha(\mathbf{r}, t) = \begin{cases} 1 & \text{for } \mathbf{r} \in dv^\alpha \\ 0 & \text{for } \mathbf{r} \in dv^\beta, \quad \beta \neq \alpha \end{cases} \quad (2.1)$$

is introduced, where t denotes the time. Thus, for the partial volume (realistic) element dv^α , with (2.1) the following relation is obtained:

$$\begin{aligned} dv^\alpha(\mathbf{x}, t) &= \int_{dv} \chi^\alpha(\mathbf{r}, t) dv_\mu \\ &= \int_{dv} \chi^\alpha(\mathbf{x} + \boldsymbol{\xi}, t) dv_\mu, \quad \mathbf{r} = \mathbf{x} + \boldsymbol{\xi}. \end{aligned} \quad (2.2)$$

Here the position vector \mathbf{r} is substituted by $\mathbf{x} + \boldsymbol{\xi}$, and the integration refers to the microscopic local $\boldsymbol{\xi}$ -reference system with the origin \mathbf{x} , i.e. the centre of dv . With (2.2) the volume fractions can be formulated as follows

$$n^\alpha(\mathbf{x}, t) = \frac{dv^\alpha}{dv} = \frac{1}{dv} \int_{dv} \chi^\alpha(\mathbf{r}, t) dv_\mu. \quad (2.3)$$

The volume fractions in (2.3) satisfy the volume fraction condition for κ constituents φ^α :

$$\sum_{\alpha=1}^{\kappa} n^\alpha = 1, \quad (2.4)$$

which can be easily proved. If the porous solid is saturated the volume fraction condition (2.4) turns out to be an important constraint in the constitutive theory.

With the help of the volume fractions the real mass of each constituent will be distributed continuously over the control space. This procedure leads to the creation of substitute continua with smooth properties and reduced densities. These substitute continua have less to do with the true constituents of the porous medium. Several extended considerations are needed, in particular, in order to formulate the boundary value problems.

2.2. Kinematics

As pointed out in Section 2.1 the substitute continua created with the help of the volume fractions show at first glance the same properties as a mixture. Concerning the kinematics of the substitute continua with imaginary material points there are two fundamental assumptions to describe the kinematics of the constituents:

- (1) Each spatial points \mathbf{x} of the actual placement is simultaneously occupied by material points X^α of all κ constituents φ^α at time t .
- (2) Each constituent is assigned an independent state of motion.

In particular, the second assumption is strictly valid only for mixtures. However, a porous medium is not a mixture. Indeed, there is interaction between the constituents in a saturated porous solid not only due to friction effects but also due to a space displacement, described by the saturation condition (2.4). It is easily recognized that this condition must always be fulfilled to maintain the saturation. The saturation condition constrains the motions of the individual constituents as will be shown in Section 3 on constitutive theory.

If the motion of a constituent is understood as a chronological succession of the placements χ_α , then for the position \mathbf{x} of the material points X^α , which are in turn described by the reference positions \mathbf{X}_α at time $t = t_0$, the following relation holds at time t :

$$\mathbf{x} = \chi_\alpha(\mathbf{X}_\alpha, t). \quad (2.5)$$

Equation (2.5) represents the Lagrangian description of the motion. The function χ_α (2.5) is postulated to be unique and uniquely invertible at any time. The existence of a function inverse to (2.5) leads to the Eulerian description of motion, i.e. :

$$\mathbf{X}_\alpha = \chi_\alpha^{-1}(\mathbf{x}, t). \quad (2.6)$$

A mathematically sufficient condition for the existence of eqn (2.6) is given, if the Jacobian

$$J_\alpha = \det \mathbf{F}_\alpha \quad (2.7)$$

differs from zero at each point \mathbf{X}_α . Here

$$\mathbf{F}_\alpha = \frac{\partial \chi_\alpha(\mathbf{X}_\alpha, t)}{\partial \mathbf{X}_\alpha} = \text{grad}_\alpha \chi_\alpha = \text{grad}_\alpha \mathbf{x} \quad (2.8)$$

is the so-called deformation gradient. From the Lagrangian description of the motion function χ_α in (2.5) the velocity and the acceleration of a material point of a constituent φ^α can be computed :

$$\mathbf{x}'_\alpha = \frac{\partial \chi_\alpha(\mathbf{X}_\alpha, t)}{\partial t}, \quad \mathbf{x}''_\alpha = \frac{\partial^2 \chi_\alpha(\mathbf{X}_\alpha, t)}{\partial t^2}. \quad (2.9)_1$$

Using (2.6), it follows the Eulerian description of the velocity and the acceleration :

$$\mathbf{x}'_\alpha = \mathbf{x}'_\alpha(\mathbf{x}, t), \quad \mathbf{x}''_\alpha = \mathbf{x}''_\alpha(\mathbf{x}, t). \quad (2.9)_2$$

As the velocity of a material point is defined in (2.9)₁ and (2.9)₂, the material time derivative of a differentiable function $\Gamma(\mathbf{x}, t)$ can be formulated :

$$\Gamma'_\alpha = \frac{\partial \Gamma}{\partial t} + \text{grad } \Gamma \cdot \mathbf{x}'_\alpha. \quad (2.10)$$

Here, the material time derivative is shown for a scalar function. The definition can easily be extended to vector- and tensor-value functions.

In the following, some deformation measurements are needed. The basic quantity for these measurements is the deformation gradient (2.8). For our purpose it is convenient to introduce only the right Cauchy–Green deformation tensor

$$\mathbf{C}_\alpha = \mathbf{F}_\alpha^T \mathbf{F}_\alpha. \tag{2.11}$$

Moreover, from (2.9)₂ the velocity gradient \mathbf{L}_α with respect to the actual placement can be calculated which is also related to the deformation gradient and the material time derivative of the deformation gradient

$$\mathbf{L}_\alpha = \text{grad } \mathbf{x}'_\alpha, \quad \mathbf{L}_\alpha = (\mathbf{F}_\alpha)'_\alpha \mathbf{F}_\alpha^{-1}. \tag{2.12}$$

The symmetric part of the velocity gradient \mathbf{L}_α is given by

$$\mathbf{D}_\alpha = \frac{1}{2}(\mathbf{L}_\alpha + \mathbf{L}_\alpha^T). \tag{2.13}$$

Moreover, two additional time derivatives are required for the following investigations, namely the time derivatives of the Jacobian (2.7) and of the right Cauchy–Green tensor (2.11)

$$(J_\alpha)'_\alpha = J_\alpha(\mathbf{D}_\alpha \cdot \mathbf{I}), \quad (\mathbf{C}_\alpha)'_\alpha = 2\mathbf{F}_\alpha^T \mathbf{D}_\alpha \mathbf{F}_\alpha. \tag{2.14}$$

For further investigations two important multiplicative decompositions of the deformation gradient (2.8) are necessary. The first decomposition concerns the split into volume-preserving and spherical parts denoted by the symbols $(\bar{\cdot})$ and $(\overset{\circ}{\cdot})$:

$$\mathbf{F}_\alpha = \overset{\circ}{\mathbf{F}}_\alpha \bar{\mathbf{F}}_\alpha, \tag{2.15}$$

where

$$\overset{\circ}{\mathbf{F}}_\alpha = (J_\alpha)^{1/3} \mathbf{I} \tag{2.16}$$

is the spherical part of \mathbf{F}_α . Moreover, it is recognized that

$$\det \bar{\mathbf{F}}_\alpha = 1 \tag{2.17}$$

and that $\bar{\mathbf{F}}_\alpha$ is the volume-preserving part of \mathbf{F}_α .

The second decomposition of the deformation gradient \mathbf{F}_α is motivated by the observation that the deformations of porous media are composed of the deformations of the real material and the changes of the pores. Therefore it is assumed that the deformation gradient can be multiplicatively decomposed into a macroscopic tensor $\mathbf{F}_{\alpha R}$ which describes the deformations at the microscale, namely of the real materials, and into a tensor $\mathbf{F}_{\alpha N}$ which reflects the remaining part of the deformation of the control space (an extensive discussion of this decomposition is contained in the two papers by Bluhm and de Boer, 1994; de Boer, 1995a):

$$\mathbf{F}_\alpha = \mathbf{F}_{\alpha N} \mathbf{F}_{\alpha R}. \tag{2.18}$$

The usefulness of the above decomposition (2.18) has already been shown in the aforementioned papers, in particular, to describe properties at the microscale as, e.g., compressibility and incompressibility. For this purpose it is necessary to formulate some additional measurements using $\mathbf{F}_{\alpha R}$ and $\mathbf{F}_{\alpha N}$. This can be done according to the definitions

Table 1. Deformation measurements

\mathbf{F}_α	$\mathbf{F}_{\alpha N}$	$\mathbf{F}_{\alpha R}$
$J_\alpha = \det \mathbf{F}_\alpha$	$J_{\alpha N} = \det \mathbf{F}_{\alpha N}$	$J_{\alpha R} = \det \mathbf{F}_{\alpha R}$
$\mathbf{C}_\alpha = \mathbf{F}_\alpha^T \mathbf{F}_\alpha$	$\mathbf{C}_{\alpha N} = \mathbf{F}_{\alpha N}^T \mathbf{F}_{\alpha N}$	$\mathbf{C}_{\alpha R} = \mathbf{F}_{\alpha R}^T \mathbf{F}_{\alpha R}$
$\mathbf{L}_\alpha = (\mathbf{F}_\alpha)_\alpha' \mathbf{F}_\alpha^{-1}$	$\mathbf{L}_{\alpha N} = (\mathbf{F}_{\alpha N})_\alpha' \mathbf{F}_{\alpha N}^{-1}$	$\mathbf{L}_{\alpha R} = (\mathbf{F}_{\alpha R})_\alpha' \mathbf{F}_{\alpha R}^{-1}$
$\mathbf{D}_\alpha = \frac{1}{2}(\mathbf{L}_\alpha + \mathbf{L}_\alpha^T)$	$\mathbf{D}_{\alpha N} = \frac{1}{2}(\mathbf{L}_{\alpha N} + \mathbf{L}_{\alpha N}^T)$	$\mathbf{D}_{\alpha R} = \frac{1}{2}(\mathbf{L}_{\alpha R} + \mathbf{L}_{\alpha R}^T)$
$(J_\alpha)_\alpha' = J_\alpha (\mathbf{D}_\alpha \cdot \mathbf{I})$	$(J_{\alpha N})_\alpha' = J_{\alpha N} (\mathbf{D}_{\alpha N} \cdot \mathbf{I})$	$(J_{\alpha R})_\alpha' = J_{\alpha R} (\mathbf{D}_{\alpha R} \cdot \mathbf{I})$
$(\mathbf{C}_\alpha)_\alpha' = 2\mathbf{F}_\alpha^T \mathbf{D}_\alpha \mathbf{F}_\alpha$	$(\mathbf{C}_{\alpha N})_\alpha' = 2\mathbf{F}_{\alpha N}^T \mathbf{D}_{\alpha N} \mathbf{F}_{\alpha N}$	$(\mathbf{C}_{\alpha R})_\alpha' = 2\mathbf{F}_{\alpha R}^T \mathbf{D}_{\alpha R} \mathbf{F}_{\alpha R}$
$\mathbf{F}_\alpha = \hat{\mathbf{F}}_\alpha \bar{\mathbf{F}}_\alpha$	$\mathbf{F}_{\alpha N} = \hat{\mathbf{F}}_{\alpha N} \bar{\mathbf{F}}_{\alpha N}$	$\mathbf{F}_{\alpha R} = \hat{\mathbf{F}}_{\alpha R} \bar{\mathbf{F}}_{\alpha R}$
$J_\alpha = \det \bar{\mathbf{F}}_\alpha = 1$	$J_{\alpha N} = \det \bar{\mathbf{F}}_{\alpha N} = 1$	$J_{\alpha R} = \det \bar{\mathbf{F}}_{\alpha R} = 1$
$\bar{\mathbf{J}}_\alpha = \mathbf{J}_\alpha$	$\bar{\mathbf{J}}_{\alpha N} = \mathbf{J}_{\alpha N}$	$\bar{\mathbf{J}}_{\alpha R} = \mathbf{J}_{\alpha R}$
$\bar{\mathbf{C}}_\alpha = (J_\alpha)^{2/3} \bar{\mathbf{C}}_\alpha$	$\bar{\mathbf{C}}_{\alpha N} = (J_{\alpha N})^{2/3} \bar{\mathbf{C}}_{\alpha N}$	$\bar{\mathbf{C}}_{\alpha R} = (J_{\alpha R})^{2/3} \bar{\mathbf{C}}_{\alpha R}$
$\bar{\mathbf{C}}_\alpha = \bar{\mathbf{F}}_\alpha^T \bar{\mathbf{F}}_\alpha$	$\bar{\mathbf{C}}_{\alpha N} = \bar{\mathbf{F}}_{\alpha N}^T \bar{\mathbf{F}}_{\alpha N}$	$\bar{\mathbf{C}}_{\alpha R} = \bar{\mathbf{F}}_{\alpha R}^T \bar{\mathbf{F}}_{\alpha R}$
$(J_\alpha)_\alpha' = (J_\alpha)_\alpha'$	$(J_{\alpha N})_\alpha' = (J_{\alpha N})_\alpha'$	$(J_{\alpha R})_\alpha' = (J_{\alpha R})_\alpha'$
$(\bar{\mathbf{C}}_\alpha)_\alpha' = 2\bar{\mathbf{F}}_\alpha^T \bar{\mathbf{D}}_\alpha^D \bar{\mathbf{F}}_\alpha$	$(\bar{\mathbf{C}}_{\alpha N})_\alpha' = 2\bar{\mathbf{F}}_{\alpha N}^T \bar{\mathbf{D}}_{\alpha N}^D \bar{\mathbf{F}}_{\alpha N}$	$(\bar{\mathbf{C}}_{\alpha R})_\alpha' = 2\bar{\mathbf{F}}_{\alpha R}^T \bar{\mathbf{D}}_{\alpha R}^D \bar{\mathbf{F}}_{\alpha R}$

$\bar{\mathbf{D}}_\alpha^D$ is the deviator of \mathbf{D}_α .

and the derivatives in this section. The results are summarized in Table 1 to follow where, also the results for the partial constituent φ^α are represented.

For further investigations the volume elements in the actual and reference placements must be considered. It is well-known in continuum mechanics that the following transport theorem concerning the volume elements holds:

$$dv = J_\alpha dv_{0\alpha} \tag{2.19}$$

where

$$dv_{0\alpha} = dv_{0\alpha}(t = t_0), \quad dv = dv(\mathbf{x}, t) \tag{2.20}$$

are the volume elements in the reference placement at every position \mathbf{X}_α , denoted by the lower index α and in the actual placement at the position \mathbf{x} , respectively. With the relation (2.19) it is easy to formulate the volume strain e_α which is important for the investigations to follow:

$$e_\alpha = \frac{dv - dv_{0\alpha}}{dv_{0\alpha}} = \frac{dv}{dv_{0\alpha}} - 1 = J_\alpha - 1, \tag{2.21}$$

where (2.19) has been used.

In a similar way the volume strain of real materials can be calculated (see de Boer, 1995):

$$e_{\alpha R} = J_{\alpha R} - 1. \tag{2.22}$$

Thus, the Jacobian $J_{\alpha R}$ is related to the volume strain $e_{\alpha R}$:

$$J_{\alpha R} = 1 + e_{\alpha R}, \tag{2.23}$$

whereas the Jacobian $J_{\alpha N}$ is related to the volume fractions (see de Boer, 1995a):

$$J_{\alpha N} = \frac{n_{0\alpha}^\alpha}{n^\alpha}. \tag{2.24}$$

Here $n_{0\alpha}^\alpha$ is the volume fraction in the reference placement.

After the extensive discussions of the kinematics of a saturated porous medium the balance equations will be briefly reviewed.

2.3. Balance equations

In order to derive the balance equations of the mixture theory, balance equations for each constituent φ^α must be established, where interaction or supply terms, respectively, are used for the description of chemical and physical exchange processes. The sum of the partial balances of the κ constituents φ^α results in the balance laws of the mixture, which must be equivalent to the balance laws of a one-component material. This requirement involves constraints for the introduced interaction terms (For a complete derivation, see Bluhm, 1996).

2.3.1. *Balance of mass* With respect to the volume element dv , the balance equation of mass for each constituent φ^α is the local statement

$$(\rho^\alpha)'_\alpha + \rho^\alpha \operatorname{div} \mathbf{x}'_\alpha = \hat{\rho}^\alpha, \quad (2.25)$$

where ρ^α is the partial density of φ^α and $\hat{\rho}^\alpha$ represents the mass supplied to φ^α by all other constituents that occupy \mathbf{x} at time t , which must meet the demand

$$\sum_{\alpha=1}^{\kappa} \hat{\rho}^\alpha = 0, \quad (2.26)$$

i.e., the sum of the local mass supply terms of all κ constituents vanishes.

2.3.2. *Balance of momentum* In the local statement, the axiom of the balance of linear momentum for a constituent φ^α is

$$\operatorname{div} \mathbf{T}^\alpha + \rho^\alpha (\mathbf{b}^\alpha - \mathbf{x}''_\alpha) + \hat{\mathbf{p}}^\alpha = \hat{\rho}^\alpha \mathbf{x}'_\alpha, \quad (2.27)$$

where \mathbf{T}^α represents the partial Cauchy stress tensor of the constituent φ^α and $\rho^\alpha \mathbf{b}^\alpha$ is the external body force density. The quantity $\hat{\mathbf{p}}^\alpha$ denotes the linear momentum supply per unit volume which can be interpreted as an interaction force and is constrained by

$$\sum_{\alpha=1}^{\kappa} \hat{\mathbf{p}}^\alpha = \mathbf{0}, \quad (2.28)$$

i.e., the sum of all interaction forces of all constituents φ^α vanishes.

2.3.3. *Balance of moment of momentum* As we are not interested in the investigations of polar materials, we do not consider any moment of momentum exchange between the individual constituents. Then the local statement of the balance equation of moment of momentum yields (see de Boer *et al.*, 1991)

$$\mathbf{I} \times \mathbf{T}^\alpha = \mathbf{0}. \quad (2.29)$$

The cross product, connecting the identity tensor \mathbf{I} and Cauchy's stress tensor \mathbf{T}^α is explained in de Boer (1982). Equation (2.29) is only fulfilled if the stress tensor \mathbf{T}^α is symmetric. Thus,

$$\mathbf{T}^\alpha = (\mathbf{T}^\alpha)^T. \quad (2.30)$$

In the case of existing moment of momentum supplies the balance equation of moment in momentum yields the non-symmetry of \mathbf{T}^α . This implies in continuum mechanics the introduction of additional rotational degrees of freedom for the material point.

2.3.4. *Balance of energy* The axiom of the energy balance of the constituent φ^α is introduced in the following local form

$$\rho^\alpha (\varepsilon^\alpha)'_\alpha - \mathbf{T}^\alpha \cdot \mathbf{D}_\alpha - \rho^\alpha r^\alpha - \operatorname{div} \mathbf{q}^\alpha = \hat{e}_\alpha - \hat{\rho}^\alpha (\varepsilon^\alpha - \frac{1}{2} \mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha). \quad (2.31)$$

Herein ε^α stands for the partial specific internal energy of φ^α , r^α is the partial external heat supply and \mathbf{q}^α is the heat flux vector. The quantity \hat{e}^α represents the local energy supply to φ^α from the other $\kappa - 1$ constituents and is constrained by

$$\sum_{\alpha=1}^{\kappa} \hat{e}^\alpha = 0. \quad (2.32)$$

In order to derive the constraints on the supply terms (2.26), (2.28), and (2.32) some convenient summations must be introduced to avoid some obscure consequences of the derivations of the barycentric velocity (see de Boer, 1995b; Bluhm, 1996).

2.3.5. The entropy inequality A useful tool to gain restrictions for constitutive response functions is the entropy inequality. This relation follows from the balance of energy after some manipulations with the absolute temperature. It can be shown (see e.g. Planck, 1964) that the absolute temperature can serve as an integrating factor for the sum of the rate of internal energy and of the stress power (see also Müller, 1979). This expression, the above stated sum divided by the absolute temperature, is denoted as entropy and thermodynamics is based on this notion—introduced by Clausius (1865). However, this conception seems only to be clear for reversible processes. For non-reversible processes the whole procedure to construct a fundamental inequality remains in some parts obscure, mysterious and incomprehensible (see Balian, 1982). Moreover, Baierlein (1992) criticized the entropy idea with the following words: “Students find entropy a mysterious concept—and not surprisingly so, for it is a difficult notion”.

As no improvement in the formulation of the second law of thermodynamics is in sight we will introduce the second law of thermodynamics in the usual way more so as in the constitutive theory the entropy inequality yields good results. In the mixture theory the postulate of one common entropy inequality for the whole mixture body is simultaneously a necessary and a sufficient condition for the existence of dissipation mechanisms within the mixture. If all constituents φ^α have the same absolute Kelvin temperature Θ the derivation of the entropy inequality results in (see de Boer *et al.*, 1991):

$$\sum_{\alpha=1}^{\kappa} \left[-\rho^\alpha (\psi^\alpha)'_\alpha - \Theta'_\alpha \rho^\alpha \eta^\alpha - \hat{\rho}^\alpha \cdot \mathbf{x}'_\alpha + \mathbf{T}^\alpha \cdot \mathbf{D}_\alpha + \frac{1}{\Theta} \mathbf{q}^\alpha \cdot \operatorname{grad} \Theta - \hat{\rho}^\alpha \left(\psi^\alpha - \frac{1}{2} \mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha \right) \right] \geq 0, \quad (2.33)$$

where $\psi^\alpha = \psi^\alpha(\mathbf{x}, t)$ is the free Helmholtz energy and $\eta^\alpha = \eta^\alpha(\mathbf{x}, t)$ the specific entropy. The free Helmholtz energy function is related to the specific energy and the specific entropy by

$$\psi^\alpha = \varepsilon^\alpha - \Theta \eta^\alpha. \quad (2.34)$$

It should be mentioned that in the case of isothermal deformations and elastic material behaviour the free Helmholtz energy function turns into the specific stored energy function.

In Section 2 the general thermodynamic structure of fluid-saturated compressible and incompressible porous solids is developed. In particular, the multiplicative decomposition of the deformation gradient can throw light on some properties of the microscale and the entropy inequality can yield important restrictions for the constitutive equations, not only for the stress and interaction forces but also for phases transitions. However, the last point will be investigated in another context. In the next section on the constitutive theory we restrict our treatment to a binary elastic porous medium.

3. CONSTITUTIVE THEORY

3.1. Preliminaries

It is easy to recognize from the preceding developed field equations that not all mechanical and thermodynamical fields can be determined by the balance equations if the body force density and the external heat supply are known. Therefore, constitutive equations are needed for the unknown fields that connect these quantities with motion and temperature functions and derivations of these functions. This means that in the porous media theory constitutive equations must be formulated for partial and true quantities as well as for several coupling mechanisms between the different constituents.

The development of constitutive equations for saturated porous media is based on the same principles that govern the derivation of constitutive equations in classical continuum mechanics of one-component materials. These are: determinism, equipresence, local action, material objectivity, and dissipation.

Guided by these principles, constitutive equations for a binary model, namely compressible or incompressible elastic porous solids, filled with an incompressible fluid, will be derived in the following.

3.2. Constraints and entropy inequality

Before going into the details, the saturation condition (2.4) for a binary model ($S =$ solid, $F =$ fluid)

$$n^S + n^F = 1 \quad (3.1)$$

will be examined. The material time derivative of (3.1) yields

$$(n^S)'_S + (n^F)'_F - \text{grad } n^F \cdot (\mathbf{x}'_F - \mathbf{x}'_S) = 0. \quad (3.2)$$

Considering (2.24) and some time derivative contained in Table 1 the saturation condition in the rate formulation is gained:

$$-n^S(\mathbf{D}_S \cdot \mathbf{I}) + n^S(\mathbf{D}_{SR} \cdot \mathbf{I}) - n^F(\mathbf{D}_F \cdot \mathbf{I}) + n^F(\mathbf{D}_{FR} \cdot \mathbf{I}) - \text{grad } n^F \cdot (\mathbf{x}'_F - \mathbf{x}'_S) = 0, \quad (3.3)$$

where in the course of the evaluation $(\mathbf{D}_{SN} \cdot \mathbf{I})$ and $(\mathbf{D}_{FN} \cdot \mathbf{I})$ have been substituted by the differences $(\mathbf{D}_S \cdot \mathbf{I})$ minus $(\mathbf{D}_{SR} \cdot \mathbf{I})$ and $(\mathbf{D}_F \cdot \mathbf{I})$ minus $(\mathbf{D}_{FR} \cdot \mathbf{I})$.

Relation (3.3) clearly shows that the volume changes of the solid and fluid phases are coupled. Thus, the motions are constrained.

This fact has some consequences. If the entropy inequality or another similar inequality is used to gain restrictions for the constitutive equations one has to be careful in the evaluation as the volume changes are coupled. The entropy inequality can be considered as a minimum problem which reaches its minimum at zero. Thus, a constraint like the saturation condition must be added in the evaluation process to the entropy inequality provided with a Lagrangian factor. From the mathematical point of view the saturation condition (3.3) must be an equation in excess. Therefore, the system of field equations of the two phases must be constructed in such a way that this requirement is satisfied. This can be done by introducing constitutive equations for the Jacobian J_{SR} or more general for the right Cauchy–Green deformation tensor \mathbf{C}_{SR} in order to incorporate the whole deformation of the solid phases at the microlevel, see Bluhm (1996) and de Boer (1995a). Then, the system of fields is closed and the saturation condition (3.3) is a surplus equation. For more details concerning the closure problem the reader is referred to Bluhm (1996) and de Boer (1995a).

It should be emphasized that the saturation condition (3.3) must always be considered as a constraint independent of the material behaviour.

As aforementioned constitutive equations must be formulated for the right Cauchy–Green deformation tensor \mathbf{bC}_{SR} because no balance equation for this quantity is contained in the mixture theory. The constitutive equation for \mathbf{C}_{SR} must reflect some properties of the

microscale and bring these to the macroscale. In the following a rather simple relation is chosen :

$$\mathbf{C}_{SR} = \mathbf{C}_{SR}(\mathbf{C}_S). \tag{3.4}$$

More complicated ansätze for \mathbf{C}_{SR} are currently under study.

With the material time derivatives in Table 1 we obtain considering (3.4) :

$$\mathbf{D}_{SR} \cdot \mathbf{I} = \mathcal{F}_S \cdot \mathbf{D}_S, \tag{3.5}$$

where

$$\mathcal{F}_S = \mathbf{F}_S \left(\frac{\partial \mathbf{C}_{SR}}{\partial \mathbf{C}_S} \right)^T \mathbf{C}_{SR}^{-1} \mathbf{F}_S^T. \tag{3.6}$$

Thus, with (3.5) the constraint (3.3) can be reformulated as :

$$n^S(\mathbf{I} - \mathcal{F}_S) \cdot \mathbf{D}_S + n^F(\mathbf{D}_F \cdot \mathbf{I}) - n^F(\mathbf{D}_{FR} \cdot \mathbf{I}) + \text{grad } n^F \cdot (\mathbf{x}'_F - \mathbf{x}'_S) = 0. \tag{3.7}$$

This constraint as an equation in excess in the system of fields must be provided with a Lagrange multiplier λ and added to the entropy inequality (2.33) which is treated as a minimum problem for the fields. For the proposed binary model the following inequality for isothermal deformations is obtained from (2.33) considering (3.7) and (2.28) :

$$-\rho^S(\psi^S)'_S - \rho^F(\psi^F)'_F + \mathbf{D}_S \cdot [\mathbf{T}^S + n^S \lambda (\mathbf{I} - \mathcal{F}_S)] + \mathbf{D}_F \cdot [\mathbf{T}^F + n^F \lambda \mathbf{I}] - n^F(\mathbf{D}_{FR} \cdot \mathbf{I}) - (\hat{\mathbf{p}}^F - \lambda \text{grad } n^F) \cdot (\mathbf{x}'_F - \mathbf{x}'_S) \geq 0. \tag{3.8}$$

In (3.8) any mass exchange between the constituents is neglected.

A major simplification of the constitutive theory is given by the incompressibility conditions of the constituents. In particular, for liquid saturated porous solids this condition is a reasonable assumption, as liquids are incompressible in a wide stress-range and the compressibility of the realistic solid material can be neglected in many cases in comparison with the compressibility of the bulk body. The incompressibility conditions are :

$$J_{SR} = 1, \quad J_{FR} = 1, \tag{3.9}$$

or in the rate formulation

$$n^S \mathbf{D}_{SR} \cdot \mathbf{I} = 0, \quad n^F \mathbf{D}_{FR} \cdot \mathbf{I} = 0. \tag{3.10}$$

The rate formulations of the incompressibility conditions must be multiplied by the volume fractions n^S and n^F because they are referred to an intermediate placement (for more details see Bluhm and Boer, 1994). Moreover, the conditions (3.10) provided with Lagrangian multipliers κ^{SR} and κ^{FR} must be added to the entropy inequality (2.33) along with the saturation condition (3.3) multiplied by the Lagrangian multiplier λ .

From (2.33) the following inequality for the proposed incompressible model for isothermal deformations is obtained, where the mass exchange between the constituents is neglected.

$$-\rho^S(\psi^S)'_S - \rho^F(\psi^F)'_F + \mathbf{D}_S \cdot [\mathbf{T}^S - n^S \lambda \mathbf{I} - n^S(\lambda - \kappa^{SR}) \mathcal{F}_S] + \mathbf{D}_F \cdot (\mathbf{T}^F + n^F \lambda \mathbf{I}) - n^F(\mathbf{D}_{FR} \cdot \mathbf{I})(\lambda - \kappa^{FR}) - (\hat{\mathbf{p}}^F - \lambda \text{grad } n^F) \cdot (\mathbf{x}'_F - \mathbf{x}'_S) \geq 0. \tag{3.11}$$

The hybrid model, i.e. compressible solid and incompressible fluid phases can easily be

derived combining the above stated inequalities. However, in all cases one has to recall the closure problem. If, e.g., the solid material is incompressible the incompressibility condition must be incorporated as a constraint, but no additional constitutive equation for a solid quantity should be formulated. This would be an additional equation in excess which violates the closure problem stated above.

At the end of this section some remarks are concerned with the principle of determinism. This principle is the leading principle in the constitutive theory (see Truesdell and Noll, 1965). The principle of determinism demands that “the stress in a body is determined by the history of the motion of that body”. However, both aforementioned inequalities indicate that the principle of determinism must obviously be modified. Indeed this was already done by Truesdell and Noll (1965):

“Principle of determinism for simple materials subject to internal constraint : The stress \mathbf{T} at time t is determined by the history $\mathbf{F}^{(t)}(s)$ of the deformation gradient only to within a stress \mathbf{N} that does no work in any motion satisfying the constraints.”

The above principle of determinism including the effect of internal constraints was, however, formulated for a one-component continuum. For a saturated porous medium it must be extended. It influences not only the stress states of the individual constituents, but also the interaction forces, see eqns (3.8) and (3.11).

3.3. Evaluation of the entropy inequality

With the entropy inequalities (3.9) and (3.11) the basic relations are developed to gain restrictions for the constitutive equations for the binary model under study. In the following these restrictions will be obtained for porous media models with compressible and incompressible constituents.

Constitutive equations (response functions) are necessary for saturated porous models with:

- (1) Incompressible elastic solid and incompressible fluid materials

$$\mathcal{R}_I := \{\psi^\alpha, \mathbf{T}_{EI}^\alpha, \hat{\mathbf{p}}_E^\alpha\}, \tag{3.12}$$

where

$$\begin{aligned} \mathbf{T}_{EI}^S &= \mathbf{T}^S + n^S \lambda \mathbf{I} - n^S (\lambda - \kappa^{SR}) \mathcal{F}_S, \\ \mathbf{T}_{EI}^F &= \mathbf{T}^F + n^F \lambda \mathbf{I}. \end{aligned} \tag{3.13}$$

- (2) Compressible elastic solid and incompressible fluid materials (hybrid model of first type)

$$\mathcal{R}_{H1} := \{\psi^\alpha, \mathbf{T}_{EC}^S, \mathbf{T}_{EI}^F, \hat{\mathbf{p}}_E^F\}; \tag{3.14}_1$$

where

$$\mathbf{T}_{EC}^S = \mathbf{T}^S + n^S \lambda (\mathbf{I} - \mathcal{F}_S). \tag{3.14}_2$$

The principles of determinism and local action state that the response functions in (3.12) and (3.14) for a material point \mathbf{X}_x , at any time t , and any place \mathbf{x} , must be determined by the history of an arbitrary small neighbourhood of this material point. The history of this small neighbourhood is laid down by the process variables $s(\mathbf{x}, t)$ which reflect the history of the motions of the two partial constituents solid and fluid. For the elastic model under study, the following process variables are postulated:

$$s := \{\mathbf{C}_S, \mathbf{x}'_F - \mathbf{x}'_S\}. \tag{3.15}$$

As the incorporation of the reference placements \mathbf{X}_x of the constituents φ^α is neglected,

only homogeneous porous media can be described. The choice of the process variables is determined as follows: the elastic deformations of the partial porous solid is described by \mathbf{C}_S . The velocity difference $\mathbf{x}'_F - \mathbf{x}'_S$ governs dissipative effects.

For the aforementioned models (point 1–2), thermodynamical restrictions result from the entropy inequalities (3.8), (3.11), for compressible and incompressible models (point 1 and 2). Using standard arguments with several symmetry and skew symmetry conditions (de Boer, 1993; de Boer and Kowalski, 1995) the investigations yield the models to be governed by constitutive equations which are relatively simple. The whole evaluation of the different forms of the entropy inequality leads to a lengthy formalism which would exceed the frame of this paper. Thus, only the main results for the individual models are listed.

3.3.1. *Incompressible model (incompressible elastic solid matrix saturated by an incompressible fluid)*. This model has been repeatedly treated in the literature and the model, neglecting all deformations at the microscale, is in general accepted. The evaluation of the entropy inequality in the form (3.11) yields the following restrictions for the constitutive equations:

$$\mathbf{T}^S = -n^S \lambda \mathbf{I} + n^S (\lambda - \kappa^{SR}) \bar{\mathcal{F}}_S + \mathbf{T}_{EI}^S \quad (3.16)$$

with the following effective stresses for the solid and fluid phases

$$\mathbf{T}_{EI}^S = 2\rho^S \mathbf{F}_S \frac{\partial \psi^S}{\partial \mathbf{C}_S} \mathbf{F}_S^T + \rho^S \frac{\partial \psi^S}{\partial (\mathbf{x}'_F - \mathbf{x}'_S)} \otimes (\mathbf{x}'_F - \mathbf{x}'_S), \quad (3.17)$$

$$\mathbf{T}^F = -n^F \lambda \mathbf{I} + \mathbf{T}_{EI}^F, \quad (3.18)$$

$$\mathbf{T}_{EI}^F = \rho^F \frac{\partial \psi^F}{\partial (\mathbf{x}'_F - \mathbf{x}'_S)} \otimes (\mathbf{x}'_F - \mathbf{x}'_S). \quad (3.19)$$

Moreover, it turns out that

$$\lambda = \kappa^{FR}. \quad (3.20)$$

As κ^{FR} is the Lagrange multiplier related to the incompressibility condition, κ^{FR} can be identified as the pore-liquid pressure p . Thus

$$\lambda = p. \quad (3.21)$$

Furthermore, a dissipation inequality remains. With (3.21) this inequality reads as

$$(\hat{\mathbf{p}}^F - p \text{grad } n^F) \cdot (\mathbf{x}'_F - \mathbf{x}'_S) \geq 0. \quad (3.22)$$

Now, the mixture equilibrium is examined. The interaction force $\hat{\mathbf{p}}^F$ is given by

$$\hat{\mathbf{p}} = p \text{grad } n^F, \quad (3.23)$$

and the constitutive relation for the stress of the fluid phase simplifies to

$$\mathbf{T}^F = -n^F p \mathbf{I}. \quad (3.24)$$

With (3.21) the constitutive equation for \mathbf{T}^S takes the following form in the mixture equilibrium state:

$$\mathbf{T}^S = -n^S p \mathbf{I} + n^S (p - \kappa^{SR}) \bar{\mathcal{F}}_S + 2\rho^S \mathbf{F}_S \frac{\partial \psi^S}{\partial \mathbf{C}_S} \mathbf{F}_S^T. \quad (3.25)$$

If the deformations at the microscale $\bar{\mathcal{F}}_S$, which represent the volume-preserving part of the micromechanical deformations, are neglected, the form of the constitutive equation remains as already stated by Fillunger (1936). In this case the Lagrange multipliers are equal

$$\lambda = \kappa^{FR} = \kappa^{SR} = p. \quad (3.26)$$

The above stated constitutive equations define a porous media model which plays an important role in many branches of engineering, e.g. in soil mechanics and in the suspension theory.

3.3.2. *Hybrid model of first type (compressible elastic solid matrix saturated by an incompressible liquid)*. This model has been a subject for several experimental investigations (see, e.g., Lade and de Boer, 1994) because of its importance, for example, in rock mechanics. The main results of the thermodynamical investigations in the mixture equilibrium state are:

$$\mathbf{T}^S = -n^S p (\mathbf{I} - \bar{\mathcal{F}}_S) + \mathbf{T}_{EC}^S, \quad \mathbf{T}_{EC}^S = 2\rho^S \mathbf{F}_S \frac{\partial \psi^S}{\partial \mathbf{C}_S} \mathbf{F}_S^T, \quad (3.27)$$

$$\mathbf{T}^F = -n^F p \mathbf{I}, \quad (3.28)$$

$$\hat{\mathbf{p}}^F = p \text{grad } n^F. \quad (3.29)$$

The constitutive equations (3.27) and (3.28) contain Suklje's (1969) effective stress principle which represents an improvement of von Terzaghi's effective stress principle. For a purely hydrostatic stress state from (3.27)

$$p^S = -n^S p \left(1 - J_{SN} \frac{\partial J_{SR}}{\partial J_S} \right) + p_E^S \quad (3.30)$$

is obtained, where the influence of $\bar{\mathbf{C}}_{SR}$ has been neglected.

With the constitutive assumption

$$J_{SR} = J_{S^{SR}/C_S}, \quad (3.31)$$

where C_{SR} and C_S are the compressibilities of the true solid material and of the partial solid body. With (3.31) the constitutive relation for the hydrostatic stress (3.30) yields

$$p^S = -n^S p \left(1 - \frac{C_{SR}}{C_S} \right) + p_E^S. \quad (3.32)$$

Suklje's (1969) formula is concerned with the total hydrostatic stress of the mixture body. The sum of (3.28) and (3.32) yields

$$\bar{p} = p^F + p^S = -p \left(1 - n^S \frac{C_{SR}}{C_S} \right) + p_E^S, \quad (3.33)$$

thus, Suklje's (1969) formula is derived (see also the extensive discussion of the effective

stress principle in Lade and de Boer, 1994). The formula (3.33) fits the test observation of Nur and Byerlee (1971) very well (see also de Boer, 1995a).

3.4. Complementary remarks

The complementary remarks concern an empty porous solid, the determination of the interface pressure (Lagrange multiplier) λ and the determination of uplift, friction and capillarity.

It can easily be proved that for empty porous solids the volume fraction condition (2.4), although still valid, is not an equation in excess. Therefore, it is not a constraint and does not enter the entropy inequality. Thus, the stresses in the partial bodies are identical with the effective stresses of the solid phase in (3.17) or (3.27).

The interface pressure p cannot be determined by a constitutive equation. Rather, we must use the balance equation of motion to compute p . In the mixture equilibrium state

$$\mathbf{T}^F = -n^F p \mathbf{I} \quad (3.34)$$

is valid, see (3.24). Moreover, in the mixture equilibrium state the interaction force $\hat{\mathbf{p}}^F$ can be expressed by the pore pressure, see (3.23):

$$\hat{\mathbf{p}}^F = p \text{grad } n^F. \quad (3.35)$$

With (3.34) and (3.35) and the acceleration of gravity \mathbf{g} the balance equation of momentum (2.27) turns out to be

$$-\text{grad}(n^F p) + \rho^F (\mathbf{g} - \mathbf{x}_F'') + p \text{grad } n^F = \mathbf{0} \quad (3.36)$$

or in the static state

$$-n^F \text{grad } p + \rho^F \mathbf{g} = \mathbf{0}. \quad (3.37)$$

Introducing the average density of the true fluid ρ^{FR} the relation

$$\rho^F = n^F \rho^{FR} \quad (3.38)$$

holds. Moreover, choosing the vertical coordinate z and the corresponding unit vector \mathbf{e}_z so that

$$\mathbf{g} = g \mathbf{e}_z \quad (3.39)$$

from (3.36) the relation

$$\frac{\partial p}{\partial z} = \rho^{FR} g \quad (3.40)$$

is obtained considering (3.37) and (3.38). From (3.36) the explicit value for the pore-fluid pressure is gained

$$p = p_0 + \rho^{FR} g z,$$

or

$$p = p_0 + \gamma^{FR} z, \quad (3.41)$$

with p_0 as a possible pore-fluid pressure for $z = 0$ and γ^{FR} as the specific weight of the pore-fluid.

The last complementary remark is concerned with the phenomena uplift, friction, and capillarity—a binary model with compressible and incompressible constituents. From the balance equation of momentum (2.27) excluding any mass exchange the equations of motion for the mixture body in the gravity field is gained

$$\operatorname{div}(\mathbf{T}^S + \mathbf{T}^F) + (\rho^S + \rho^F)\mathbf{g} - \rho^S \mathbf{x}_S'' - \rho^F \mathbf{x}_F'' = \mathbf{0}, \quad (3.42)$$

where \mathbf{g} is the acceleration of gravity. With the constitutive relations (3.20) and (3.22) for the saturated binary model as well as (3.54) and the corresponding relation for the solid, eqn (3.59) turns into

$$\operatorname{div} \mathbf{T}_{EC}^S - \operatorname{div} \{p(\mathbf{I} + n^S \mathcal{F}_S)\} + (n^S \gamma^{SR} + n^F \gamma^{FR}) \frac{\mathbf{g}}{|\mathbf{g}|} - \rho^S \mathbf{x}_S'' - \rho^F \mathbf{x}_F'' = \mathbf{0} \quad (3.43)$$

with

$$\gamma^{\alpha\beta} = \rho^{\alpha R} |\mathbf{g}| \quad (3.61)$$

as the specific weight. Considering the saturation condition (2.4) the following relation holds:

$$n^S \gamma^{SR} + n^F \gamma^{FR} = \gamma^{FR} + n^S (\gamma^{SR} - \gamma^{FR}). \quad (3.45)$$

Along with relation (3.45) we derive from (3.43) the following equation:

$$\operatorname{div} \mathbf{T}_{EC}^S - \operatorname{div} \{p(\mathbf{I} + n^S \mathcal{F}_S)\} + \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|} + n^S (\gamma^{SR} - \gamma^{FR}) \frac{\mathbf{g}}{|\mathbf{g}|} - \rho^S \mathbf{x}_S'' - \rho^F \mathbf{x}_F'' = \mathbf{0}. \quad (3.46)$$

In (3.46)

$$\begin{aligned} \mathbf{k}_u &= -n^S \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|} \\ &= (n^F - 1) \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|} \end{aligned} \quad (3.47)$$

is the uplift force (see de Boer and Ehlers, 1990). The derivation of the uplift formula reveals that the uplift phenomenon is independent of such properties as compressibility and incompressibility. This is not surprising as (3.47) can be derived from Archimedes' principle which refers only to the density of the fluid and the displaced amount of fluid.

Moreover, the vector

$$\mathbf{k}_f = \gamma^{FR} \frac{\mathbf{g}}{|\mathbf{g}|} - \operatorname{grad} p \quad (3.48)$$

in (3.46) can be interpreted as the friction force in the flow zone or the capillary force in the suction zone, see de Boer and Ehlers (1990).

4. CONCLUSIONS

In this paper the general thermodynamic structure to gain restrictions for fluid-saturated compressible and incompressible porous solids has been developed. In particular, the saturation condition in the rate formulation has been incorporated into the thermodynamics structure as a constraint. Whereas the consideration of incompressible material in the development of the saturation condition is relatively simple the consideration of compressible material is more complicated.

In this case the multiplicative decomposition of the deformation gradient makes the derivation of the rate formulation of the saturation condition much easier. Moreover, with the multiplicative decomposition of the deformation gradient the incompressibility conditions for the true materials can be clearly stated by kinematic quantities and it is not necessary to use material dependent quantities, namely the true densities, to describe incompressibility.

The thermodynamics structure developed in this paper yields important restrictions for the effective Cauchy's stress tensors of the individual constituents consisting of incompressible and compressible materials. Furthermore, the thermodynamic investigations yield reasonable constitutive equations for the interaction force in a binary model. It is revealed that the restrictions gained from the entropy inequality are also only valid for the effective interaction force which is not influenced by compressibility or incompressibility.

A new approach to develop consistent mathematical models with incompressible and/or compressible phases is currently under study at Essen University.

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